On optimal investment for a behavioural investor in multiperiod incomplete market models

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Abstract

We provide easily verifiable conditions for the well-posedness of the optimal investment problem for a behavioral investor in an incomplete discrete-time multiperiod financial market model, for the first time in the literature. Under suitable assumptions we also establish the existence of optimal strategies.

Keyword : Optimisation, existence and well-posedness in behavioral finance, "S-shaped" utility function, distorsion, Choquet integral.

1 Introduction

A classical optimization problem of mathematical finance is to find the investment strategy that maximizes the expected von Neumann-Morgenstern utility of the portfolio value of some economic agent, see e.g. Chapter 2 of Föllmer and Schied [2002]. In mathematical terms, $Eu(X)$ needs to be maximized in X where u is a concave increasing function and X runs over values of admissible portfolios.

In the paper Kahneman and Tversky [1979], based on experimentation, the authors contest the expected utility paradigm and propose the cumulative prospect theory. This theory asserts that: first, agents behave differently on gains and on losses. Second, agents overweight small probabilities and underweight large probabilities. Third, agents evaluate assets in comparison with some benchmark rather than based on final wealth positions. This can be translated into mathematics by the following assumptions: investors use an "S-shaped" utility function u (i.e. $u(x) = u_+(x)$, $x \geq 0$; $u(x) = -u_-(-x)$, $x < 0$ where $u_+, u_- : \mathbb{R}_+ \to \mathbb{R}$ are concave and increasing) and they also distort the probability measure: instead of expectations, Choquet integrals appear. Furthermore, maximization of their objective function takes place over the random variables $X - B$ where B is a fixed (stochastic) reference point and X, again, runs over the values of admissible portfolios.

That paper triggered an avalanche of subsequent investigations, especially in the economics literature. See the references of Jin and Zhou [2008] for a sample relevant to our present discussions. Mathematical treatments rarely went beyond simple, one-step models (we only know of Berkelaar et al. [2004], Jin and Zhou [2008] and Carlier and Dana [2011] where multiperiod models are studied; in Berkelaar et al. [2004], however, no probability distortion is considered). The reason for this, as pointed out in Jin and Zhou [2008], is the presence of massively difficult obstacles: the objective function is non-concave and the probability distortions make it impossible to use dynamic programming and the related machinery based on the Bellmann equation.

In Jin and Zhou [2008] a rather specific continuous-time market model (driven by Brownian motion) is treated. As this model is complete (all "reasonable" random variables can be realized by continuous trading), the behavioral investment problem can be reduced to a (still very difficult) static optimization over a set of random variables.

The arguments of both Jin and Zhou [2008] and Carlier and Dana [2011] heavily rely on completeness and there has not yet been any treatment of discrete-time multiperiod models in the literature. In the present article we make a substantial step ahead by studying this problem in a multiperiod, generically incomplete market model. We allow for a possibly stochastic reference point B and need no concavity assumptions on u_+, u_-: only their behavior at infinity matters.

The issue of well-posedness is a recurrent theme in this literature (see Bernard and Ghossoub. [2010], He and Zhou). As far as we know, our Theorem 4.3 below is the first result of its kind for discrete-time multiperiod models.

The conditions in Jin and Zhou [2008] are not easily checked and have no obvious interpretations. Here we manage to provide intuitive and easily verifiable conditions which apply in the case where u_+, u_- and the probability distortions are all "power-like" functions satisfying certain parameter constraints and where appropriate moment conditions hold for the price process. We also give examples providing parameter restrictions which are necessary for well-posedness.

Existence of optimal strategies is fairly subtle in this setting. Assuming a certain structure for the information flow (see Assumption 5.1) we manage to establish an existence result in Theorem 5.6 below. Finally, we exhibit examples showing that our assumptions are satisfied in a reasonably large class of models.

The paper is organized as follows: in section 2 we introduce our models; section 3 presents examples pertinent to the well-posedness of the problem; section 4 provides a sufficient condition for well-posedness in a multiperiod market; section 5 proves the existence of optimizers under appropriate conditions, section 6 gives examples while section 7 contains technical material.

2 Market model description

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a discrete-time filtered probability space with time horizon $T \in \mathbb{N}$. The set of m-dimensional $\mathcal{F}_t$-measurable random variables will be denoted by Ξ_t^m.

Let $\{S_t,\ 0 \le t \le T\}$ be a d-dimensional adapted process representing the (discounted) price of d securities in the financial market in consideration. The notation $\Delta S_t := S_t - S_{t-1}$ will often be used. Trading strategies are given by d-dimensional processes $\{\theta_t,\ 1 \le t \le T\}$ which are supposed to be predictable (i.e. θ_t is $\mathcal{F}_{t-1}$-measurable). The class of all such strategies is denoted by Φ.

Trading is assumed to be self-financing, so the value process of a strategy $\theta \in \Phi$ is

$$X_t^{z,\theta} := z + \sum_{j=1}^{t} \theta_j \Delta S_j,$$

where z is the initial capital of the agent in consideration and the concatenation xy of elements $x, y \in \mathbb{R}^d$ means that we take their scalar product.

Consider the following technical condition (R). It says, roughly speaking, that there are no redundant assets, even conditionally, see also Remark 9.1 of Föllmer and Schied [2002].

(R) *The support of the (regular) conditional distribution of ΔS_t with respect to $\mathcal{F}_{t-1}$ is not contained in any proper affine subspace of $\mathbb{R}^d$, almost surely, for all $1 \le t \le T$.*

The following absence of arbitrage condition is standard, it is equivalent to the existence of a risk-neutral measure in discrete time markets with finite horizon, see e.g. Dalang et al. [1990].

(NA) *If $X_T^{0,\theta} \ge 0$ a.s. for some $\theta \in \Phi$ then $X_T^{0,\theta} = 0$ a.s.*

The next proposition is a trivial reformulation of Proposition 1.1 in Carassus and Rásonyi [2007].

Proposition 2.1. *The condition* (R) + (NA) *is equivalent to the existence of $\mathcal{F}_t$-measurable positive random variables κ_t, π_t , $0 \le t \le T-1$ such that*

$$\text{ess.}\inf_{\xi \in \Xi_t^d} P(\xi \Delta S_{t+1} \le -\kappa_t |\xi| \,/\mathcal{F}_t) \ge \pi_t \ \ a.s.$$

Let $\mathcal{W}$ denote the set of $\mathbb{R}$-valued random variables Y such that $E|Y|^p < \infty$ for all $p > 0$. This family is clearly closed under addition, multiplication and taking conditional expectation. The family of nonnegative elements in $\mathcal{W}$ is denoted by $\mathcal{W}^+$. With a slight abuse of notation, for a d-dimensional random variable Y, we write $Y \in \mathcal{W}$ when we indeed mean $|Y| \in \mathcal{W}$. We will also need $\mathcal{W}_t^+ := \mathcal{W}^+ \cap \Xi_t^1$.

We now present the hypotheses on the market model that will be needed for our main results in the sequel.

Assumption 2.2. *For all $t \ge 1$, $\Delta S_t \in \mathcal{W}$. Furthermore, for $0 \le t \le T-1$, there exist $\kappa_t, \pi_t \in \Xi_t^1$ satisfying $1/\kappa_t, 1/\pi_t \in \mathcal{W}_t^+$ such that*

$$\text{ess.}\inf_{\xi \in \Xi_t^d} P(\xi \Delta S_{t+1} \le -\kappa_t |\xi| /\mathcal{F}_t) \ge \pi_t \ \ a.s. \tag{1}$$

The first item in the above assumption could be weakened to the existence of the Nth moment for N large enough but this would lead to complications with no essential gain in generality, which we prefer to avoid. In the light of Proposition 2.1, (1) is a certain strong form of no-arbitrage. Section 6 exhibits concrete examples showing that Assumption 2.2 holds in a broad class of market models. We note that, by Proposition 2.1, Assumption 2.2 implies both (NA) and (R) above.

Now we turn to investors' behavior, as modeled by cumulative prospect theory, see Kahneman and Tversky [1979], Tversky and Kahneman [1992]. Agents' attitude towards gains and losses will be expressed by the functions u_+, u_-. Agents are assumed to have a (possibly stochastic) reference point B and a distorted view of reality expressed by the probability distortion functions w_+, w_-.

Formally, we assume that $u_+, u_- : \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing, continuous functions with $u_+(0) = u_-(0) = 0$. We fix B, a scalar-valued random variable. The functions $w_+, w_- : [0,1] \to [0,1]$ are nondecreasing and continuous with $w_+(0) = w_-(0) = 0$ and $w_+(1) = w_-(1) = 1$.

Remark 2.3. Actually, the main results of the present article need less about $u_\pm, w_\pm$ than stipulated here, in particular, about continuity and monotonicity. But as these are fairly natural requirements for agents' preferences, we make these assumptions throughout the paper.

Example 2.4. A typical choice is taking

$$u_+(x) = x^\alpha, \quad u_-(x) = kx^\beta$$

for some $k > 0$ and setting

$$w_+(t) = \frac{t^\gamma}{(t^\gamma + (1-t)^\gamma)^{1/\gamma}}, \quad w_-(t) = \frac{t^\delta}{(t^\delta + (1-t)^\delta)^{1/\delta}},$$

with constants $0 < \alpha, \beta, \gamma, \delta \le 1$. In Tversky and Kahneman [1992], the following choice was made: $\alpha = \beta = 0.88$, $k = 2.25$, $\gamma = 0.61$ and $\delta = 0.69$.

We define, for $X_0 \in \Xi_0^1$ and $\theta \in \Phi$,

$$V^+(X_0; \theta_1, \ldots, \theta_T) := \int_0^\infty w_+\left(P\left(u_+\left(\left[X_T^{X_0,\theta} - B\right]_+\right) \ge y\right)\right) dy,$$

and

$$V^-(X_0; \theta_1, \ldots, \theta_T) := \int_0^\infty w_-\left(P\left(u_-\left(\left[X_T^{X_0,\theta} - B\right]_-\right) \ge y\right)\right) dy,$$

and whenever $V^-(X_0;\theta_1,\dots,\theta_T)<\infty$ we set

$$V(X_0;\theta_1,\dots,\theta_T):=V^+(X_0;\theta_1,\dots,\theta_T)-V^-(X_0;\theta_1,\dots,\theta_T).$$

We denote by $\mathcal{A}(X_0)$ the set of strategies $\theta\in\Phi$ such that

$$V^-(X_0;\theta_1,\dots,\theta_T)<\infty$$

and we call them admissible (with respect to X_0).

Remark 2.5. If there were no probability distortions (i.e. $w_+(t)=w_-(t)=t$) then we would simply get $V^+(X_0;\theta_1,\dots,\theta_T)=Eu_+\left(\left[X_T^{X_0,\theta}-B\right]_+\right)$ and $V^-(X_0;\theta_1,\dots,\theta_T)=Eu_-\left(\left[X_T^{X_0,\theta}-B\right]_-\right)$, i.e. the expected "utility" of gains (resp. losses) with respect to the given reference point B. We refer to Carassus and Pham [2009] for the explicit treatment of this problem in a continuous time, complete case under the assumptions that u_+ is concave, u_- is convex and B is deterministic.

The present paper is concerned with maximizing $V(X_0;\theta_1,\dots,\theta_T)$ over $\theta\in\mathcal{A}(X_0)$. We seek to find conditions ensuring well-posedness, i.e.

$$\sup_{\theta\in\mathcal{A}(X_0)} V(X_0;\theta_1,\dots,\theta_T)<\infty,$$

and the existence of $\theta^*\in\mathcal{A}(X_0)$ attaining this supremum.

3 A first look at well-posedness

For simplicity we assume that $u_+(x)=x^\alpha$ and $u_-(x)=x^\beta$ for some $0<\alpha,\beta\le 1$; the distortion functions are $w_+(t)=t^\gamma$, $w_-(t)=t^\delta$ for some $0<\gamma,\delta\le 1$. The example given below applies also to w_+,w_- with a power-like behavior near 0 such as those in Example 2.4 above.

Let us consider a two-step market model with $S_0=0$, ΔS_1 uniform on $[-1,1]$, $P(\Delta S_2=\pm 1)=1/2$ and ΔS_2 is independent of ΔS_1. This choice assures that there is absence of arbitrage. Let $\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2$ be the natural filtration of S_0,S_1,S_2.

Let us choose initial capital $X_0=0$ and reference point $B=0$. We consider the strategy given by $\theta_1=0$ and $\theta_2=g(\Delta S_1)$ for some measurable $g:\mathbb{R}\to\mathbb{R}$ such that the distribution function of θ_2 is

$$F(y)=0,\ y<1,\quad F(y)=1-\frac{1}{y^\ell},\ y\ge 1,$$

where $\ell>0$ will be chosen later; such a g exists as ΔS_1 has uniform law. It follows that

$$V^+(X_0;\theta_1,\theta_2)=\int_0^\infty P^\gamma((\theta_2\Delta S_2)_+^\alpha>y)=\int_1^\infty \frac{1}{2^\gamma}\frac{1}{y^{\ell\gamma/\alpha}}dy,$$

and

$$V^-(X_0;\theta_1,\theta_2)=\int_0^\infty P^\delta((\theta_2\Delta S_2)_-^\beta>y)=\int_1^\infty \frac{1}{2^\delta}\frac{1}{y^{\ell\delta/\beta}}dy.$$

If we have $\alpha/\gamma>\beta/\delta$ then there is $\ell>0$ such that

$$\frac{\gamma\ell}{\alpha}<1<\frac{\delta\ell}{\beta},$$

which entails $V^-(X_0;\theta_1,\theta_2)<\infty$ and $V^+(X_0;\theta_1,\theta_2)=\infty$ so the optimization problem becomes ill-posed.

One may wonder whether this phenomenon could be ruled out by restricting the set of strategies e.g. to bounded ones. The answer is no. Considering $\theta_1(n) := 0, \theta_2(n) := \min\{\theta_2, n\}$ for $n \in \mathbb{N}$ we obtain that $V^+(X_0; \theta_1(n), \theta_2(n)) \to \infty$, $V^-(X_0; \theta_1(n), \theta_2(n)) \to V^-(X_0; \theta_1, \theta_2) < \infty$ by monotone convergence, which shows that we still have

$$\sup_{\psi} V(X_0; \psi_1, \psi_2) = \infty,$$

where ψ ranges over the family of bounded strategies.

This shows that the ill-posedness phenomenon is not just a pathology but comes from the fact that one may use the information available at time 1 when choosing the investment strategy θ_2.

We mention another case of ill-posedness which is present already in one-step models, as noticed in He and Zhou and Bernard and Ghossoub. [2010]. We generalize a bit the previous setting, $u_+, u_-, \Delta S_1$ remain unchanged but we allow general distortions, assuming only that $w_+(t), w_-(t) > 0$ for $t > 0$. The market is defined by $S_0 = 0$, $\Delta S_2 = \pm 1$ with probabilities $p, 1-p$ for some $0 < p < 1$ and ΔS_2 is independent of ΔS_1. Take $\theta_1(n) = 0 = X_0 = B$ and $\theta_2(n) := n$, $n \in \mathbb{N}$, then $V^+(X_0; \theta_1(n), \theta_2(n)) = n^\alpha w_+(p)$ and $V^-(X_0; \theta_1(n), \theta_2(n)) = n^\beta w_-(p)$. If $\alpha > \beta$ then, *whatever* w_+, w_- *are*, we have $V(X_0; \theta_1(n), \theta_2(n)) \to \infty$. Hence, in order to get a well-posed problem one needs to have $\alpha \leq \beta$, as already observed in Bernard and Ghossoub. [2010] and He and Zhou.

We add a comment on the case $\alpha = \beta$ (the choice of Tversky and Kahneman [1992]): *whatever* w_+, w_- *are*, we may easily choose p such that the problem becomes ill-posed.

Since it would be difficult to exclude such a simple types of probability laws for ΔS_2 on economic grounds we are led to the conclusion that in order to get a mathematically meaningful optimization problem for a reasonably wide range of price processes one needs to assume both

$$\alpha < \beta \quad \text{and} \quad \alpha/\gamma \leq \beta/\delta. \tag{2}$$

In the following section we propose an easily verifiable sufficient condition for the well-posedness of this problem in multiperiod discrete-time market models. The decisive condition we require is $\alpha/\gamma < \beta$, see (7) below. This is stronger than (2) but still reasonably general. If $w_-(t) = t$ (i.e. $\delta = 1$, no distortion on loss probabilities) then (7) is essentially sharp, as the present section highlights.

4 Well-posedness in the multiperiod case

Assumptions 2.2, 4.1 and 4.2 will be in force throughout this section. We first present the conditions we need on $u_\pm, w_\pm$. Basically, we require that $u_\pm$ behave in a power-like way at infinity and $w_\pm$ do likewise in the neighborhood of 0. Condition (7) has already been mentioned in the previous section. It has a rather straightforward interpretation: the investor takes losses more seriously than gains and the distortion on gains is not too strong so that it is still outbalanced by loss aversion (as represented by parameter β). We stress that no concavity assumption is made on u_+, u_-, unlike in all related papers.

Assumption 4.1. *We assume that*

$$\begin{aligned} u_+(x) &\leq k_+ x^\alpha + \bar{k}_+, & (3) \\ k_- x^\beta - \bar{k}_- &\leq u_-(x), & (4) \\ w_+(x) &\leq g x^\gamma, & (5) \\ w_-(x) &\geq f x,, & (6) \end{aligned}$$

with $0 < \alpha, \beta, \gamma \leq 1$, $k_\pm, \bar{k}_\pm, g, f > 0$ *fixed constants and*

$$\frac{\alpha}{\gamma} < \beta. \tag{7}$$

This allows us to fix λ *such that* $\lambda\gamma > 1$ *and* $\lambda\alpha < \beta$.

We remark that the functions in Example 2.4 satisfy Assumption 4.1 whenever (7) holds.

The assumption below requires that the reference point B should be comparable to the market performance in the sense that it can be sub-hedged by some portfolio strategy $\phi \in \Phi$.

Assumption 4.2. *We fix a scalar random variable B such that, for some strategy $\phi \in \Phi$ and for some $b \in \mathbb{R}$, we have*

$$X_T^{b,\phi} = b + \sum_{t=1}^{T} \phi_t \Delta S_t \leq B. \tag{8}$$

The main result of the present section is the following.

Theorem 4.3. *Under Assumptions 2.2, 4.1 and 4.2,*

$$\sup_{\theta \in \mathcal{A}(X_0)} V(X_0; \theta_1, \ldots, \theta_T) < \infty,$$

whenever $X_0 \in \Xi_0^1$ with $E|X_0|^\beta < \infty$.

In particular, the result applies for X_0 a deterministic constant. We need to do some preparatory work before proving Theorem 4.3.

Lemma 4.4. *There exist constants $\tilde{k}, \ell, \tilde{\ell} > 0$, such that*

$$\begin{aligned} V^+(X_0; \theta_1, \ldots, \theta_T) &\leq \tilde{k} E\left(1 + |X_t + \sum_{n=1}^{T} (\theta_n - \phi_n)\Delta S_n|^{\alpha\lambda}\right) \quad (9) \\ &:= \tilde{V}^+(X_0; \theta_1, \ldots, \theta_T), \\ V^-(X_0; \theta_1, \ldots, \theta_T) &\geq \ell E\left([X_0 + \sum_{n=1}^{T} (\theta_n - \phi_n)\Delta S_n - b]_-^\beta\right) - \tilde{\ell} \quad (10) \\ &:= \tilde{V}^-(X_0; \theta_1, \ldots, \theta_T). \end{aligned}$$

Proof. We will use the facts that for $0 < \eta \leq 1$ one has

$$|x + y|^\eta \leq |x|^\eta + |y|^\eta \tag{11}$$

for all $x, y \in \mathbb{R}$ and for $1 \leq \eta$ we have $|x + y|^\eta \leq 2^{\eta-1}(|x|^\eta + |y|^\eta)$.

We get, using (5) and Chebishev's inequality:

$$V^+(X_0; \theta_1, \ldots, \theta_T) \leq 1 + g \int_1^\infty \frac{E^\gamma\left(u_+^\lambda([X_0 + \sum_{n=1}^{T} \theta_n \Delta S_n - B]_+)\right)}{y^{\lambda\gamma}} dy \tag{12}$$

Evaluating the integral and using (3) we continue the estimation as

$$\begin{aligned} V^+(X_0; \theta_1, \ldots, \theta_T) &\leq 1 + \frac{g}{\lambda\gamma - 1} E^\gamma\left(2^{\lambda-1} k_+^\lambda [X_0 + \sum_{n=1}^{T} \theta_n \Delta S_n - B]_+^{\alpha\lambda} + 2^{\lambda-1} \bar{k}_+^\lambda\right) \\ &\leq 1 + \frac{g}{\lambda\gamma - 1} \Big[k_+^\lambda 2^{\lambda-1} \left(E(|X_0 + \sum_{n=1}^{T} (\theta_n - \phi_n)\Delta S_n|^{\alpha\lambda}) + |b|^{\alpha\lambda}\right) \\ &\quad + \bar{k}_+^\lambda 2^{\lambda-1} + 1 \Big], \end{aligned}$$

using the rough estimate $x^\gamma \leq x + 1$, $x \geq 0$, Assumption 4.2 and the fact that $C_1 \geq C_2$ implies that $(Y - C_1)_+ \leq (Y - C_2)_+$. This gives the first statement. For the second inequality note that, by (6) and Assumption 4.2,

$$
\begin{aligned}
V^{-}(X_0;\theta_1,\ldots,\theta_T) &\geq f\int_0^\infty P\left(u_-([X_0+\sum_{n=1}^{T}(\theta_n-\phi_n)\Delta S_n-b]_-)\geq y\right)dy \\
&= fEu_-\left([X_0+\sum_{n=1}^{T}(\theta_n-\phi_n)\Delta S_n-b]_-\right) \\
&\geq fk_-E\left([X_0+\sum_{n=1}^{T}(\theta_n-\phi_n)\Delta S_n-b]_-^{\beta}\right)-f\bar{k}_-.
\end{aligned}
$$

□

Whenever $\tilde{V}^{-}(X_0;\theta_1,\ldots,\theta_T)<\infty$, we set

$$
\tilde{V}(X_0;\theta_1,\ldots,\theta_T):=\tilde{V}^{+}(X_0;\theta_1,\ldots,\theta_T)-\tilde{V}^{-}(X_0;\theta_1,\ldots,\theta_T).
$$

For $X_0\in\Xi_0^1$ we also introduce $\tilde{\mathcal{A}}(X_0)$ as the set of $\theta\in\Phi$ such that $\tilde{V}^{-}(X_0;\theta_1,\ldots,\theta_T)<\infty$.

For proving Theorem 4.3, we will make use of an auxiliary optimization problem with objective function $\tilde{V}(X_0;\theta_1,\ldots,\theta_T)$. As no probability distortions are involved this time, we can perform a kind of dynamic programming on this auxiliary problem. To this end we introduce, for all $t=0,\ldots,T$, $X_t\in\Xi_t^1$ and $\theta_n\in\Xi_{n-1}^d$, $n\geq t+1$ the quantities

$$
\begin{aligned}
\tilde{V}_t^{+}(X_t;\theta_{t+1},\ldots,\theta_T) &:= \tilde{k}E\left(1+|X_t+\sum_{n=t+1}^{T}(\theta_n-\phi_n)\Delta S_n|^{\alpha\lambda}/\mathcal{F}_t\right), \\
\tilde{V}_t^{-}(X_t;\theta_{t+1},\ldots,\theta_T) &:= \ell E\left([X_t+\sum_{n=t+1}^{T}(\theta_n-\phi_n)\Delta S_n-b]_-^{\beta}/\mathcal{F}_t\right)-\tilde{\ell}.
\end{aligned}
$$

Whenever $\tilde{V}_t^{-}(X_t;\theta_{t+1},\ldots,\theta_T)<\infty$ a.s., we also define

$$
\tilde{V}_t(X_t;\theta_{t+1},\ldots,\theta_T):=\tilde{V}_t^{+}(X_t;\theta_{t+1},\ldots,\theta_T)-\tilde{V}_t^{-}(X_t;\theta_{t+1},\ldots,\theta_T).
$$

We denote by $\tilde{\mathcal{A}}_t(X_t)$ the set of $(\theta_{t+1},\ldots,\theta_T)$ such that $\tilde{V}_t^{-}(X_t;\theta_{t+1},\ldots,\theta_T)<\infty$ a.s.

Remark 4.5. Clearly, $(\theta_{t+1},\ldots,\theta_T)\in\tilde{\mathcal{A}}_t(X_t)$ implies $(\theta_{t+m+1},\ldots,\theta_T)\in\tilde{\mathcal{A}}_{t+m}(X_t+\sum_{n=t+1}^{t+m}(\theta_n-\phi_n)\Delta S_n)$, for $m\geq 0$; this follows from the tower law for conditional expectations and the fact that a bounded from below and integrable random variable is almost surely finite. For the same reason, $\tilde{\mathcal{A}}(X_0)\subset\tilde{\mathcal{A}}_0(X_0)$ and, by Lemma 4.4, $\mathcal{A}(X_0)\subset\tilde{\mathcal{A}}(X_0)$.

The crux of our arguments is contained in the next result.

Lemma 4.6. *For each $0\leq t\leq T$, there exist $C_n^t\in\mathcal{W}_n^{+}$, $n=t,\ldots,T-1$ such that, for $(\theta_{t+1},\ldots,\theta_T)\in\tilde{\mathcal{A}}_t(X_t)$,*

$$
\tilde{V}_t(X_t;\theta_{t+1},\ldots,\theta_T)\leq\tilde{V}_t(X_t;\tilde{\theta}_{t+1},\ldots,\tilde{\theta}_T),
$$

where $(\tilde{\theta}_{t+1},\ldots,\tilde{\theta}_T)\in\tilde{\mathcal{A}}_t(X_t)$ satisfies

$$
|\tilde{\theta}_n-\phi_n|\leq C_{n-1}^t[|X_t|+1], \tag{13}
$$

for $n=t+1,\ldots,T$, whenever $X_t\in\Xi_t^1$.

Proof. Notice that for $t = T$ the statement of the Lemma is trivial as there are no strategies involved. Let us assume that the Lemma is true for $t+1$, we will deduce that it holds true for t, too. Let $X_t \in \Xi_t^1$ and $(\theta_{t+1}, \dots, \theta_T) \in \tilde{\mathcal{A}}_t(X_t)$. Let $X_{t+1} = X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}$, then $X_{t+1} \in \Xi_{t+1}^1$ and by Remark 4.5, $(\theta_{t+2}, \dots, \theta_T) \in \tilde{\mathcal{A}}_{t+1}(X_{t+1})$. By induction hypothesis take $(\hat{\theta}_{t+2}, \dots, \hat{\theta}_T) \in \tilde{\mathcal{A}}_{t+1}(X_{t+1})$ such that

$$|\hat{\theta}_n - \phi_n| \le C_{n-1}^{t+1}[|X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}| + 1] \tag{14}$$

and $\tilde{V}_{t+1}(X_{t+1}; \theta_{t+2}, \dots, \theta_T) \le \tilde{V}_{t+1}(X_{t+1}; \hat{\theta}_{t+2}, \dots, \hat{\theta}_T)$. It is clear from (14) that

$$\left| \sum_{n=t+2}^{T} (\hat{\theta}_n - \phi_n)\Delta S_n \right| \le H\left(|X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}| + 1\right)$$

for $H = \sum_{n=t+2}^{T} C_{n-1}^{t+1}|\Delta S_n| \in \mathcal{W}^+$. We have

$$\begin{aligned} \tilde{V}_t(X_t; \theta_{t+1}, \dots, \theta_T) &= E(\tilde{V}_{t+1}(X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}; \theta_{t+2}, \dots, \theta_T)/\mathcal{F}_t) \\ &\le E(\tilde{V}_{t+1}(X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}; \hat{\theta}_{t+2}, \dots, \hat{\theta}_T)/\mathcal{F}_t) \\ &= \tilde{V}_t(X_t; \theta_{t+1}, \hat{\theta}_{t+2}, \dots, \hat{\theta}_T). \end{aligned} \tag{15}$$

Fix some $\alpha\lambda < \chi < \beta$, we continue the estimation of $\tilde{V}_t^+ = \tilde{V}_t^+(X_t; \theta_{t+1}, \hat{\theta}_{t+2}, \dots, \hat{\theta}_T)$ using the (conditional) Hölder inequality for $q = \chi/(\alpha\lambda)$ and $1/p + 1/q = 1$.

$$\begin{aligned} \tilde{V}_t^+ &\le \tilde{k}\left[1 + E(|X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}|^{\alpha\lambda}/\mathcal{F}_t) + \right. \\ &\quad \left. E(H^{\alpha\lambda}|X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}|^{\alpha\lambda} + H^{\alpha\lambda}/\mathcal{F}_t)\right] \\ &\le \tilde{k}\Big[1 + |X_t|^{\alpha\lambda} + |\theta_{t+1} - \phi_{t+1}|^{\alpha\lambda} E(|\Delta S_{t+1}|^{\alpha\lambda}/\mathcal{F}_t) + E^{1/p}(H^{\alpha\lambda p}/\mathcal{F}_t)\,(\\ &\quad E^{1/q}(|X_t|^{\chi}/\mathcal{F}_t) + E^{1/q}(|\theta_{t+1} - \phi_{t+1}|^{\chi}|\Delta S_{t+1}|^{\chi}/\mathcal{F}_t)\Big) + E(H^{\alpha\lambda}/\mathcal{F}_t)\Big] \\ &\le \tilde{k}\Big[1 + |X_t|^{\alpha\lambda} + |\theta_{t+1} - \phi_{t+1}|^{\alpha\lambda} E(|\Delta S_{t+1}|^{\alpha\lambda}/\mathcal{F}_t) + E^{1/p}(H^{\alpha\lambda p}/\mathcal{F}_t)\,\Big(|X_t|^{\alpha\lambda} \\ &\quad + |\theta_{t+1} - \phi_{t+1}|^{\alpha\lambda} E^{1/q}(|\Delta S_{t+1}|^{\chi}/\mathcal{F}_t)\Big) + E(H^{\alpha\lambda}/\mathcal{F}_t)\Big]. \end{aligned}$$

It follows that, for an appropriate H_t in $\mathcal{W}_t^+$,

$$\begin{aligned} \tilde{V}_t(X_t; \theta_{t+1}, \dots, \theta_T) &\le \tilde{\ell} + H_t\left(1 + |X_t|^{\alpha\lambda} + |\theta_{t+1} - \phi_{t+1}|^{\alpha\lambda}\right) - \\ &\quad \ell E\left([X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1} + \sum_{n=t+2}^{T} (\hat{\theta}_n - \phi_n)\Delta S_n - b]_-^{\beta}/\mathcal{F}_t\right). \end{aligned} \tag{16}$$

By Lemma 4.7, the event

$$A := \{(\hat{\theta}_n - \phi_n)\Delta S_n \le 0,\, n \ge t+2;\ (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1} \le -\kappa_t|\theta_{t+1} - \phi_{t+1}|\}$$

satisfies $P(A|\mathcal{F}_t) \ge \tilde{\pi}_t$ with $1/\tilde{\pi}_t \in \mathcal{W}_t^+$, hence considering

$$F := \left\{ \frac{|\theta_{t+1} - \phi_{t+1}|\kappa_t}{2} \ge |X_t| + |b| \right\} \tag{17}$$

we have (recall that $X_{t+1} = X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}$),

$$\begin{aligned} 1_F E\left([X_{t+1} + \sum_{n=t+2}^{T} (\hat{\theta}_n - \phi_n)\Delta S_n - b]_-^{\beta}/\mathcal{F}_t\right) &\ge 1_F E\left(1_A\left(\frac{|\theta_{t+1} - \phi_{t+1}|\kappa_t}{2}\right)^{\beta}/\mathcal{F}_t\right) \\ &\ge \left(\frac{|\theta_{t+1} - \phi_{t+1}|\kappa_t}{2}\right)^{\beta} \tilde{\pi}_t 1_F. \end{aligned} \tag{18}$$

As a little digression we estimate

$$\begin{aligned}\tilde{V}_t(X_t;\phi_{t+1},\dots,\phi_T) &= E\left(\tilde{k}(1+|X_t|^{\alpha\lambda})-\ell[X_t-b]_-^{\beta}/\mathcal{F}_t\right)+\tilde{\ell}\\ &\geq -\ell|X_t|^{\beta}-\ell|b|^{\beta}. \qquad (19)\end{aligned}$$

Let us now choose the $\mathcal{F}_t$-measurable random variable C_t^t so large that on the event

$$\tilde{F}:=\{|\theta_{t+1}-\phi_{t+1}|>C_t^t[|X_t|+1]\}$$

we have

$$\begin{aligned}\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2} &\geq |X_t|+|b| \quad \text{(that is, } \tilde{F}\subset F \text{ holds)}\\ \frac{\ell}{3}\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t &\geq (\ell+H_t)|X_t|^{\beta},\\ \frac{\ell}{3}\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t &\geq 2H_t+\ell|b|^{\beta}+\tilde{\ell},\\ \frac{\ell}{3}\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t &\geq H_t|\theta_{t+1}-\phi_{t+1}|^{\alpha\lambda}.\end{aligned}$$

One can easily check that such a C_t^t exists and is in $\mathcal{W}_t^+$. On $\tilde{F}$ we have, using $|X_t|^{\alpha\lambda}\leq|X_t|^{\beta}+1$ and thanks to (16), (18) and (19):

$$\begin{aligned}\tilde{V}_t(X_t;\theta_{t+1},\dots,\theta_T)-\tilde{V}_t(X_t;\phi_{t+1},\dots,\phi_T) &\leq \tilde{\ell}+H_t\left(2+|X_t|^{\beta}+|\theta_{t+1}-\phi_{t+1}|^{\alpha\lambda}\right)-\\ &\quad \ell\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t+\ell|X_t|^{\beta}+\ell|b|^{\beta}\\ &\leq (\ell+H_t)|X_t|^{\beta}+H_t|\theta_{t+1}-\phi_{t+1}|^{\alpha\lambda}+2H_t\\ &\quad +\ell|b|^{\beta}-\ell\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t+\tilde{\ell}\\ &\leq 0. \qquad (20)\end{aligned}$$

Consequently, defining

$$\begin{aligned}\tilde{\theta}_{t+1} &:= \phi_{t+1}1_{\tilde{F}}+\theta_{t+1}1_{\tilde{F}^c},\\ \tilde{\theta}_n &:= \phi_n 1_{\tilde{F}}+\hat{\theta}_n 1_{\tilde{F}^c}, \quad n=t+2,\dots,T,\end{aligned}$$

we have, using (15) and (20),

$$\tilde{V}_t(X_t;\theta_{t+1},\dots,\theta_T)\leq\tilde{V}_t(X_t;\tilde{\theta}_{t+1},\dots,\tilde{\theta}_T)\text{ a.s..}$$

By construction,

$$|\tilde{\theta}_{t+1}-\phi_{t+1}|\leq C_t^t[|X_t|+1],$$

and, for $n\geq t+2$,

$$|\tilde{\theta}_n-\phi_n|\leq C_{n-1}^{t+1}[|X_t+(\tilde{\theta}_{t+1}-\phi_{t+1})\Delta S_{t+1}|+1]\leq C_{n-1}^{t+1}[|X_t|+C_t^t(|X_t|+1)|\Delta S_{t+1}|+1],$$

hence we may set

$$C_{n-1}^t:=C_{n-1}^{t+1}(C_t^t|\Delta S_{t+1}|+1)$$

for $n\geq t+2$. Clearly, $C_{n-1}^t\in\mathcal{W}_{n-1}^+$. To conclude the proof it remains to check that $(\tilde{\theta}_{t+1},\dots,\tilde{\theta}_T)\in\tilde{\mathcal{A}}_t(X_t)$. As by hypothesis $(\theta_{t+1},\dots,\theta_T)\in\tilde{\mathcal{A}}_t(X_t)$, we get from (15) that $\tilde{V}_t^-(X_t;\theta_t,\hat{\theta}_{t+2},\dots,\hat{\theta}_T)<\infty$. Finally,

$$\tilde{V}_t^-(X_t;\tilde{\theta}_{t+1},\dots,\tilde{\theta}_T)=1_{\tilde{F}}\ell\left((X_t-b)_-^{\beta}-\tilde{\ell}\right)+1_{\tilde{F}^c}\tilde{V}_t^-(X_t;\theta_{t+1},\hat{\theta}_{t+2},\dots,\hat{\theta}_T)<\infty \quad \text{a.s.}$$

□

Lemma 4.7. *There exists $\tilde{\pi}_t$ with $1/\tilde{\pi}_t \in \mathcal{W}_t^+$ such that*

$$P((\theta_{t+1}-\phi_{t+1})\Delta S_{t+1} \leq -\kappa_t|\theta_{t+1}-\phi_{t+1}|,\ (\hat{\theta}_n-\phi_n)\Delta S_n \leq 0,\ n=t+2,\ldots,T/\mathcal{F}_t) \geq \tilde{\pi}_t.$$

Proof. Define the events

$$\begin{aligned} A_{t+1} &:= \{(\theta_{t+1}-\phi_{t+1})\Delta S_{t+1} \leq -\kappa_t|\theta_{t+1}-\phi_{t+1}|\}, \\ A_n &:= \{(\hat{\theta}_n-\phi_n)\Delta S_n \leq 0\}, \quad t+2 \leq n \leq T. \end{aligned}$$

We prove, by induction, that for $m \geq t+1$,

$$E(1_{A_{t+1}}\ldots 1_{A_m}/\mathcal{F}_t) \geq \tilde{\pi}_t(m) \tag{21}$$

for some $\tilde{\pi}_t(m)$ with $1/\tilde{\pi}_t(m) \in \mathcal{W}_t^+$. For $m=t+1$ this is just (1). Let us assume that (21) has been shown for $m-1$, we will establish it for m.

$$\begin{aligned} E(1_{A_m}\ldots 1_{A_{t+1}}/\mathcal{F}_t) &= E(E(1_{A_m}/\mathcal{F}_{m-1})1_{A_{m-1}}\ldots 1_{A_{t+1}}/\mathcal{F}_t) \\ &\geq E(\pi_{m-1}1_{A_{m-1}}\ldots 1_{A_{t+1}}/\mathcal{F}_t) \\ &\geq \frac{E^2(1_{A_{m-1}}\ldots 1_{A_{t+1}}/\mathcal{F}_t)}{E(1/\pi_{m-1}/\mathcal{F}_t)} \geq \frac{\tilde{\pi}_t^2(m-1)}{E(1/\pi_{m-1}/\mathcal{F}_t)} \end{aligned}$$

by the (conditional) Cauchy inequality. Here $1/\tilde{\pi}_t(m-1) \in \mathcal{W}_t^+$ by the induction hypothesis, $E(1/\pi_{m-1}|\mathcal{F}_t) \in \mathcal{W}_t^+$ (since $1/\pi_{m-1} \in \mathcal{W}^+$) and the statement follows. □

Proof of Theorem 4.3. If $\mathcal{A}(X_0)$ is empty, there is nothing to prove. Otherwise, by Remark 4.5 and Lemma 4.6, for all $n \in \{1,\ldots,T\}$, there is $C_n^0 \in \mathcal{W}_n^+$ such that for all $\theta \in \mathcal{A}(X_0)$ there exists $\tilde{\theta} \in \tilde{\mathcal{A}}_0(X_0)$ satisfying $|\tilde{\theta}_n - \phi_n| \leq C_{n-1}^0[|X_0|+1]$, $1 \leq n \leq T$ and

$$\tilde{V}_0(X_0;\theta_1,\ldots,\theta_T) \leq \tilde{V}_0(X_0;\tilde{\theta}_1,\ldots,\tilde{\theta}_T).$$

As $\theta \in \mathcal{A}(X_0)$, by Lemma 4.4

$$\begin{aligned} V(X_0;\theta_1,\ldots,\theta_T) &\leq \tilde{V}(X_0;\theta_1,\ldots,\theta_T) = E\tilde{V}_0(X_0;\theta_1,\ldots,\theta_T) \leq E\tilde{V}_0(X_0;\tilde{\theta}_1,\ldots,\tilde{\theta}_T) \\ &\leq E\tilde{V}_0^+(X_0;\tilde{\theta}_1,\ldots,\tilde{\theta}_T) \\ &\leq \tilde{k}E\left(1+|X_0|^{\alpha\lambda}+\sum_{n=1}^T|\tilde{\theta}_n-\phi_n|^{\alpha\lambda}E(|\Delta S_n|^{\alpha\lambda}/\mathcal{F}_{n-1})\right) \\ &\leq \tilde{k}E\left(\left(1+|X_0|^{\alpha\lambda}\right)\left(1+\sum_{n=1}^T(C_{n-1}^0)^{\alpha\lambda}E(|\Delta S_n|^{\alpha\lambda}/\mathcal{F}_{n-1})\right)\right) \\ &\leq \tilde{k}2^{\frac{p-1}{p}}E^{1/p}(1+|X_0|^{\beta})E^{1/q}\left(1+\sum_{n=1}^T(C_{n-1}^0)^{\alpha\lambda}E(|\Delta S_n|^{\alpha\lambda}/\mathcal{F}_{n-1})\right)^q \end{aligned} \tag{22}$$

using Hölder's inequality with $p=\beta/(\alpha\lambda)$ and its conjugate number q. As $E|X_0|^{\beta}<\infty$, we get that this expression is finite, showing the Theorem. □

Remark 4.8. It is worth contrasting Theorem 4.3 with Theorem 3.2 of Jin and Zhou [2008]. The latter states, in a continuous-time context, that in a typical (complete) Brownian market model our optimization problem is ill-posed whenever u_+ is unbounded and $w_-(t)=t$ (i.e. no distortion on losses).

This phenomenon stems from the particularity of continuous-time models where the richness of attainable payoffs leads to ill-posedness. However, in our discrete-time models the family of replicable claims is relatively small hence ill-posedness does not occur even if w_- is the identity (as long as the other assumptions of Theorem 4.3 hold).

5 Existence of optimal strategies

Throughout this section, Assumptions 2.2, 4.1, 4.2, 5.1 and 5.3 will be in force.

Assumption 5.1. *Let $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $\mathcal{G}_t = \sigma(Z_1, \dots, Z_t)$ for $1 \leq t \leq T$, where the Z_i, $i = 1, \dots, T$ are $\mathbb{R}^N$-valued independent random variables. S_0 is constant and ΔS_t is a continuous function of $(Z_1, \dots, Z_t)$, for all $t \geq 1$ (hence S_t is $\mathcal{G}_t$-adapted).*

Furthermore, $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{F}_0$, $t \geq 0$, where $\mathcal{F}_0 = \sigma(\tilde{\varepsilon})$ with $\tilde{\varepsilon}$ uniformly distributed on $[0, 1]$ and independent of $(Z_1, \dots, Z_T)$.

We may think that $\mathcal{G}_t$ contains the information available at time t (given by the observable stochastic factors Z_i, $i = 1, \dots, t$) and $\mathcal{F}_0$ provides some independent random source that we use to randomize our trading strategies (in practice, one may always generate $\tilde{\varepsilon}$ e.g. using a computer). The random variables Z_i represent the "innovation": the information surplus of $\mathcal{F}_i$ with respect to $\mathcal{F}_{i-1}$, in an independent way. For the construction of the optimal strategies we use weak convergence techniques which necessitate the additional randomness provided by $\tilde{\varepsilon}$ (the situation is somewhat analogous to the construction of a weak solution for a stochastic differential equation). Assumption 5.1 holds in many cases, see section 6 and Remark 7.8 below.

Remark 5.2. *Let us take $S_0 = 0$, $P(\Delta S_1 = 2) = 3/4$, $P(\Delta S_1 = -1) = 1/4$ and $\mathcal{G}_i$, $i = 0, 1$ the natural filtration of S. Assume, moreover, that $\mathcal{F}_0 = \sigma(\varepsilon)$ where $P(\varepsilon = 1) = 2/3$, $P(\varepsilon = -1) = 1/3$ and $\mathcal{F}_1 = \mathcal{G}_1 \vee \mathcal{F}_0$. Let $\mathcal{A}(0)$ (resp. $\mathcal{A}'(0)$) denote the family of admissible strategies from 0 initial capital which are $\mathcal{G}_t$ (resp. $\mathcal{F}_t$) predictable. Consider $u_+(x) = x^{1/4}$, $u_-(x) = x$, $w_+(p) = p^{1/2}$, $w_-(p) = p$.*

We thank Andrea Meireles for numerically checking that, somewhat surprisingly,

$$\sup_{\theta \in \mathcal{A}(0)} V(0; \theta_1) < \sup_{\theta \in \mathcal{A}'(0)} V(0; \theta_1).$$

This simple example shows that additional randomness may increase the satisfaction of the agent, hence using the randomization $\tilde{\varepsilon}$ appearing in Assumption 5.1 is, at least, reasonable.

This is in stark contrast with the case where there is no distortion present. The tower law for conditional expectations shows that in this case adding an independent random variable to $\mathcal{G}_0$ cannot increase the agent's attainable level of satisfaction.

Assumption 5.3. *The random variable B is a continuous function of $(Z_1, \dots, Z_T)$, X_0 is deterministic and $\mathcal{A}(X_0)$ is not empty.*

Remark 5.4. If B is a continuous function of $(S_0, \dots, S_T)$ then Assumption 5.1 clearly implies the first part of Assumption 5.3. If $u_-(x) \leq c(1 + x^\eta)$ for some $c \geq k_-$ and $\eta \geq \beta$, $X_0, B \in \mathcal{W}$ and $w_-(t) \leq Ct^\delta$ for some $\delta \leq 1$ and $C \geq 1$ then Lemma 7.1 implies that the strategy $\theta_t = 0$, $t = 1, \dots, T$ is in $\mathcal{A}(X_0)$, in particular, the latter set is non-empty. Actually, $\theta \in \mathcal{A}(X_0)$ whenever $\theta_t \in \mathcal{W}$, $t = 1, \dots, T$. This applies, in particular, to Example 2.4.

Remark 5.5. We may and will suppose that Z_i are bounded. This can always be achieved by replacing each coordinate Z_i^j of Z_i with $\arctan Z_i^j$ for $j = 1, \dots, N$, $i = 1, \dots, T$.

We now present our main result on the existence of an optimal strategy.

Theorem 5.6. *Let Assumptions 2.2, 4.1, 4.2, 5.1 and 5.3 hold. Then there is $\theta^* \in \mathcal{A}(X_0)$ such that*

$$V(X_0; \theta_1^*, \dots, \theta_T^*) = \sup_{\theta \in \mathcal{A}(X_0)} V(X_0; \theta_1, \dots, \theta_T) < \infty.$$

Before proving Theorem 5.6, we need to extend certain arguments of Lemma 4.6 above. We fix some $\alpha\lambda < \chi < \beta$ for what follows.

Lemma 5.7. *Fix $c \in \mathbb{R}$ and β'', β' satisfying $\chi < \beta'' < \beta' < \beta$. Assume that*

$$E\tilde{V}_t(X_t; \theta_{t+1}, \ldots, \theta_T) \geq c$$

for some $\mathcal{F}_t$-measurable X_t with $E|X_t|^{\beta'} < \infty$ and for some $(\theta_{t+1}, \ldots, \theta_T) \in \tilde{\mathcal{A}}_t(X_t)$. Then there exists K_t such that

$$E|\theta_{t+1} - \phi_{t+1}|^{\beta''} \leq K_t[E|X_t|^{\beta'} + 1],$$

where K_t does not depend on either X_t or θ.

Proof. Let $X_{t+1} := X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}$ and $(\hat{\theta}_{t+2}, \ldots, \hat{\theta}_T) \in \tilde{\mathcal{A}}_{t+1}(X_{t+1})$ such that

$$|\hat{\theta}_n - \phi_n| \leq C^{t+1}_{n-1}[|X_{t+1}| + 1],$$

for $n = t+2, \ldots, T$ and

$$\tilde{V}_{t+1}(X_{t+1}; \theta_{t+2}, \ldots, \theta_T) \leq \tilde{V}_{t+1}(X_{t+1}; \hat{\theta}_{t+2}, \ldots, \hat{\theta}_T),$$

see Lemma 4.6. We can obtain equations (15) and (16) just like in the proof of Lemma 4.6. Furthermore, using (18), we get (recall (17) for the definition of F) :

$$\begin{aligned} E\tilde{V}_t(X_t; \theta_{t+1}, \ldots, \theta_T) \quad &\leq \quad E(H_t(1 + |X_t|^{\alpha\lambda} + |\theta_{t+1} - \phi_{t+1}|^{\alpha\lambda})) \\ &\quad -\ell E\left(1_F\left(\frac{|\theta_{t+1} - \phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t\right) + \tilde{\ell}. \end{aligned} \tag{23}$$

We now push further estimations in this last equation.

We may estimate, using the Hölder inequality for $p = \beta/\beta'$ and its conjugate q,

$$E\left(1_F\left(\frac{|\theta_{t+1} - \phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t\right) \geq \frac{E^p\left(1_F\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta'}\tilde{\pi}_t^{1/p}\frac{1}{\tilde{\pi}_t^{1/p}}\right)}{E^{p/q}\left(\frac{1}{\tilde{\pi}_t^{q/p}}\right)}.$$

The denominator here will be denoted C in the sequel. By Lemma 4.7, $C < \infty$.

Now let us note the trivial fact that for random variables $X, Y \geq 0$ such that $EY^{\beta'} \geq 2EX^{\beta'}$ one has $E[1_{\{Y \geq X\}}Y^{\beta'}] \geq \frac{1}{2}EY^{\beta'}$.

It follows that if

$$E\left(\frac{|\theta_{t+1} - \phi_{t+1}|\kappa_t}{2}\right)^{\beta'} \geq 2E(|X_t| + |b|)^{\beta'} \tag{24}$$

holds true then, applying the trivial $x \leq x^p + 1$, $x \geq 0$,

$$\begin{aligned} \frac{E^p\left(1_F\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta'}\right)}{C} \quad &\geq \quad \frac{E^p\left(\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta'}\right)}{2^pC} \\ &\geq \quad \frac{E\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta'} - 1}{2^pC} = c_1E(|\theta_{t+1} - \phi_{t+1}|\kappa_t)^{\beta'} - c_2 \end{aligned}$$

with suitable $c_1, c_2 > 0$. Using again Hölder's inequality with $p = \beta'/\beta''$ and its conjugate q,

$$E(|\theta_{t+1} - \phi_{t+1}|\kappa_t)^{\beta'} \quad \geq \quad \frac{E^p|\theta_{t+1} - \phi_{t+1}|^{\beta''}}{E^{p/q}\left(\frac{1}{\kappa_t^{\beta''q}}\right)} \geq \frac{E|\theta_{t+1} - \phi_{t+1}|^{\beta''} - 1}{E^{p/q}\left(\frac{1}{\kappa_t^{\beta''q}}\right)}. \tag{25}$$

With suitable $c_1', c_2' > 0$, we get, whenever (24) holds, that

$$E\left(1_F\left(\frac{|\theta_{t+1}-\phi_{t+1}|\kappa_t}{2}\right)^{\beta}\tilde{\pi}_t\right) \quad \geq \quad c_1' E|\theta_{t+1}-\phi_{t+1}|^{\beta''} - c_2'. \tag{26}$$

Estimate also, with $p := \chi/(\alpha\lambda)$,

$$\begin{aligned} E\left(H_t(1+|X_t|^{\lambda\alpha}+|\theta_{t+1}-\phi_{t+1}|^{\lambda\alpha})\right) &\leq E^{1/q}[H_t^q][1+E^{1/p}|X_t|^{\chi}+E^{1/p}|\theta_{t+1}-\phi_{t+1}|^{\chi}] \\ &\leq E^{1/q}[H_t^q][3+E|X_t|^{\chi}+E|\theta_{t+1}-\phi_{t+1}|^{\chi}] \\ &\leq \tilde{c}[1+E|X_t|^{\beta'}+E|\theta_{t+1}-\phi_{t+1}|^{\chi}], \end{aligned} \tag{27}$$

with some $\tilde{c} > 0$, using that $x^{\chi} \leq x^{\beta'}+1$, $x^{1/p} \leq x+1$, for $x \geq 0$. Furthermore, Hölder's inequality with $p = \beta''/\chi$ gives

$$E|\theta_{t+1}-\phi_{t+1}|^{\chi} \leq E^{\chi/\beta''}|\theta_{t+1}-\phi_{t+1}|^{\beta''}.$$

It follows that whenever

$$\left(E|\theta_{t+1}-\phi_{t+1}|^{\beta''}\right)^{1-\chi/\beta''} \geq \frac{2\tilde{c}}{c_1'\ell}, \tag{28}$$

one also has

$$\tilde{c}E|\theta_{t+1}-\phi_{t+1}|^{\chi} \leq \frac{c_1'\ell}{2}E|\theta_{t+1}-\phi_{t+1}|^{\beta''}. \tag{29}$$

Finally consider the condition

$$\frac{c_1'\ell}{2}E|\theta_{t+1}-\phi_{t+1}|^{\beta''} \geq \tilde{c}[1+E|X_t|^{\beta'}]+(c_2'\ell-c+1)+\tilde{\ell}. \tag{30}$$

It is easy to see that we can find some K_t, large enough, such that $E|\theta_{t+1}-\phi_{t+1}|^{\beta''} \geq K_t[E|X_t|^{\beta'}+1]$ implies that (24) (recall (25)), (28), (30) all hold true. So in this case we have, from (23), (27), (29), (26) and (30),

$$\begin{aligned} E\tilde{V}_t(X_t;\theta_{t+1},\ldots,\theta_T) &\leq \tilde{c}[1+E|X_t|^{\beta'}]+\frac{c_1'\ell}{2}E|\theta_{t+1}-\phi_{t+1}|^{\beta''} \\ &\quad -c_1'\ell E|\theta_{t+1}-\phi_{t+1}|^{\beta''}+c_2'\ell+\tilde{\ell} \\ &\leq -(c_2'\ell-c+1)+c_2'\ell < c. \end{aligned}$$

From this the statement of the lemma follows. □

Corollary 5.8. *Fix $c \in \mathbb{R}$ and assume that*

$$\tilde{V}(X_0;\theta_1,\ldots,\theta_T) = E\tilde{V}_0(X_0;\theta_1,\ldots,\theta_T) \geq c$$

for some $X_0 \in \Xi_0^1$ with $E|X_0|^{\beta} < \infty$ and some $\theta \in \tilde{\mathcal{A}}(X_0)$. Fix β_T with $\chi < \beta_T < \beta$. There exist constants $G_t, t = 0,\ldots,T-1$ such that $E|\theta_{t+1}-\phi_{t+1}|^{\beta_T} \leq G_t[E|X_0|^{\beta}+1]$ for $t = 0,\ldots,T-1$, and G_t do not depend on X_0 or on θ.

Proof. Take $\beta_T < \beta_{T-1} < \ldots < \beta_1 < \beta_0 := \beta$. We first prove, by induction on t, that $X_t := X_0 + \sum_{j=1}^{t}(\theta_j-\phi_j)\Delta S_j$, $t \geq 0$ satisfy

$$E|X_t|^{\beta_t} \leq C_t[E|X_0|^{\beta}+1],$$

for suitable $C_t > 0$. For $t = 0$ this is trivial. Assuming it for t we will show it for $t+1$. We first remark that

$$E\tilde{V}_t(X_t;\theta_{t+1},\ldots,\theta_T) = E\tilde{V}_0(X_0;\theta_1,\ldots,\theta_T) \geq c$$

and that by the induction hypothesis $E|X_t|^{\beta_t} < \infty$ holds. As $\theta \in \tilde{\mathcal{A}}(X_0) \subset \tilde{\mathcal{A}}_0(X_0)$, $(\theta_{t+1}, \ldots, \theta_T) \in \tilde{\mathcal{A}}_t(X_t)$ (see Remark 4.5). Thus Lemma 5.7 applies with the choice $\beta'' := (\beta_{t+1}+\beta_t)/2$ and $\beta' := \beta_t$, and we can estimate, using Hölder's inequality with $p := \beta''/\beta_{t+1}$ (and its conjugate number q), plugging in the induction hypothesis:

$$\begin{aligned}
E|X_{t+1}|^{\beta_{t+1}} &= E|X_t + (\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}|^{\beta_{t+1}} \\
&\leq E|X_t|^{\beta_{t+1}} + E|(\theta_{t+1} - \phi_{t+1})\Delta S_{t+1}|^{\beta_{t+1}} \\
&\leq E|X_t|^{\beta_t} + 1 + E^{1/p}|(\theta_{t+1} - \phi_{t+1})|^{\beta''} E^{1/q}|\Delta S_{t+1}|^{\beta_{t+1}q} \\
&\leq E|X_t|^{\beta_t} + 1 + C\left(E|(\theta_{t+1} - \phi_{t+1})|^{\beta''} + 1\right) \\
&\leq E|X_t|^{\beta_t} + 1 + C\left(K_t(E|X_t|^{\beta'} + 1) + 1\right) \\
&= E|X_t|^{\beta_t} + 1 + C\left(K_t(E|X_t|^{\beta_t} + 1) + 1\right) \\
&\leq (1 + CK_t)C_t\left(E|X_0|^{\beta} + 1\right) + 1 + C + CK_t
\end{aligned}$$

with $C := E^{1/q}|\Delta S_{t+1}|^{q\beta_{t+1}}$, this proves the induction hypothesis for $t+1$.

Now let us observe that, by Lemma 5.7 (with $\beta'' = \beta_{t+1}, \beta' = \beta_t$),

$$\begin{aligned}
E|\theta_{t+1} - \phi_{t+1}|^{\beta_T} &\leq E|\theta_{t+1} - \phi_{t+1}|^{\beta_{t+1}} + 1 \\
&\leq K_t[E|X_t|^{\beta_t} + 1] + 1 \leq K_t[C_t(E|X_0|^{\beta} + 1) + 1] + 1,
\end{aligned}$$

concluding the proof. □

Proof of Theorem 5.6. Lemma 7.2 with the choice $E := \tilde{\varepsilon}$, $l = 2$ gives us $\varepsilon, \varepsilon'$ independent, uniformly distributed on $[0,1]$ and $\mathcal{F}_0$-measurable. Introduce

$$\mathcal{A}'(X_0) := \{\theta \in \mathcal{A}(X_0) : \theta_t \text{ is } \mathcal{F}'_{t-1}\text{-measurable for all } t = 1, \ldots, T\},$$

where $\mathcal{F}'_t := \mathcal{G}_t \vee \sigma(\varepsilon')$. Note that if $\theta \in \mathcal{A}(X_0)$ then there exists $\theta' \in \mathcal{A}'(X_0)$ such that the law of $(\theta, \Delta S)$ equals that of $(\theta', \Delta S)$ (since the law of $\tilde{\varepsilon}$ equals that of ε' and both are independent of ΔS). It follows that for all $\theta \in \mathcal{A}(X_0)$ there is $\theta' \in \mathcal{A}(X_0)$ with

$$V(X_0; \theta_1, \ldots, \theta_T) = V(X_0; \theta'_1, \ldots, \theta'_T).$$

Take $\theta(j) \in \mathcal{A}(X_0), j \in \mathbb{N}$ such that

$$V(X_0; \theta_1(j), \ldots, \theta_T(j)) \to \sup_{\theta \in \mathcal{A}(X_0)} V(X_0; \theta_1, \ldots, \theta_T), \quad j \to \infty.$$

By Assumption 5.3 and Theorem 4.3, the supremum is finite and we can fix c such that $-\infty < c < \inf_j V(X_0; \theta_1(j), \ldots, \theta_T(j))$. By Lemma 4.4 it implies that for all j,

$$\tilde{V}(X_0; \theta_1(j), \ldots, \theta_T(j)) > c.$$

By the discussions above we may and will assume $\theta(j) \in \mathcal{A}'(X_0), j \in \mathbb{N}$. Apply Corollary 5.8 for some β_T such that $\chi < \beta_T < \beta$ to get

$$\sup_{j,t} E|\theta_t(j) - \phi_t|^{\beta_T} < \infty.$$

It follows that the sequence of $T(d+N)+1$-dimensional random variables

$$\tilde{Y}_j := (\varepsilon', \theta_1(j) - \phi_1, \ldots, \theta_T(j) - \phi_T, Z_1, \ldots, Z_T)$$

are bounded in L_{β_T} (recall Remark 5.5) and hence

$$P(|\tilde{Y}_j| > N) \leq \frac{E|\tilde{Y}_j|^{\beta_T}}{N^{\beta_T}} \leq \frac{C}{N^{\beta_T}},$$

for some fixed $C > 0$, showing that the sequence of the laws of $\tilde{Y}_j$ is tight. Then, clearly, the sequence of laws of

$$Y_j := (\varepsilon', \theta_1(j), \dots, \theta_T(j), Z_1, \dots, Z_T),$$

is also tight and hence admits a subsequence (which we continue to denote by j) weakly convergent to some probability law μ on $\mathcal{B}(\mathbb{R}^{T(d+N)+1})$.

We will construct, inductively, θ_t^*, $t = 1, \dots, T$ such that $(\varepsilon', \theta_1^*, \dots, \theta_T^*, Z_1, \dots, Z_T)$ has law μ and θ^* is $\mathcal{F}$-predictable. Let M be a $T(d+N)+1$-dimensional random variable with law μ. First note that $(M^{1+Td+1}, \dots, M^{1+Td+N})$ has the same law as Z_1,
$\dots$, $(M^{1+Td+(T-1)N+1}, \dots, M^{1+Td+TN})$ has the same law as Z_T.

Now let μ_k be the law of $(M^1, \dots, M^{1+kd}, M^{1+dT+1}, \dots, M^{1+dT+NT})$ on $\mathbb{R}^{kd+NT+1}$ (which represents the marginal of μ with respect to its first $1+kd$ and last NT coordinates), $k \geq 0$.

As a first step, we apply Lemma 7.2 with $E := \varepsilon$, $l := T$ to get $\sigma(\varepsilon)$-measurable random variables $\varepsilon_1, \dots, \varepsilon_T$ that are independent with uniform law on $[0,1]$.

Applying Lemma 7.3 with the choice $N_1 = d$, $N_2 = 1$, $Y = \varepsilon'$ and $E = \varepsilon_1$ we get a function G such that $(\varepsilon', G(\varepsilon', \varepsilon_1))$ has the same law as the marginal of μ_1 with respect to its first $1+d$ coordinates. Let Q, Q', U, U' random variables such that Q and Q' have same law and U and U' have same law. If Q is independent of U and Q' is independent of U', then (Q,U) and (Q',U') have same law.

Let $Q = (M^1, \dots, M^{d+1})$, $Q' = (\varepsilon', G(\varepsilon', \varepsilon_1))$, $U = (M^{1+dT+1}, \dots, M^{1+dT+dN})$ and $U' = (Z_1, \dots, Z_T)$. As $(\varepsilon_1, \varepsilon', Z_1, \dots, Z_T)$ are independent, we get that Q' is independent of U'. Now remark that weak convergence preserves independence, so as $(\varepsilon', \theta_1(j))$ and U' are independent for all j, we get that Q is independent of U. So we conclude that $(\varepsilon', G(\varepsilon', \varepsilon_1), Z_1, \dots, Z_T)$ has law μ_1. Define $\theta_1^* := G(\varepsilon', \varepsilon_1)$, this is clearly $\mathcal{F}_0$-measurable.

Carrying on, let us assume that we have found θ_j^*, $j = 1, \dots, k$ such that $(\varepsilon', \theta_1^*, \dots, \theta_k^*, Z_1, \dots Z_T)$ has law μ_k and θ_j^* is a function of $\varepsilon', Z_1, \dots, Z_{j-1}$, $\varepsilon_1, \dots, \varepsilon_j$ only (and is thus $\mathcal{F}_{j-1}$-measurable). We apply Lemma 7.3 with $N_1 = d$, $N_2 = kd + kN + 1$, $E = \varepsilon_{k+1}$ and

$$Y = (\varepsilon', \theta_1^*, \dots, \theta_k^*, Z_1, \dots, Z_k)$$

to get G such that $(Y, G(Y, \varepsilon_{k+1}))$ has the same law as $(M^1, \dots, M^{1+kd}, M^{1+Td+1}, \dots, M^{1+Td+kN}$, $M^{1+kd+1}, \dots, M^{1+(k+1)d})$. Thus

$$Q' = (\varepsilon', \theta_1^*, \dots, \theta_k^*, G(Y, \varepsilon_{k+1}), Z_1, \dots, Z_k)$$

has the same law as $Q = (M^1, \dots, M^{1+(k+1)d}, M^{1+Td+1}, \dots, M^{1+Td+kN})$. Now choose $U = (M^{1+dT+kN+1}, \dots, M^{1+dT+dN})$ and $U' = (Z_{k+1}, \dots, Z_T)$. As Q' depends only on $(\varepsilon_1, \dots, \varepsilon_{k+1}, \varepsilon'$, $Z_1, \dots, Z_k)$, which is independent from $(Z_{k+1}, \dots, Z_T)$, Q' is independent of U'. Moreover, $(\varepsilon', \theta_1(j), \dots, \theta_{k+1}(j)$, $Z_1, \dots, Z_k)$ and $(Z_{k+1}, \dots, Z_T)$ are independent for all j and weak convergence preserves independence, so Q is independent of U. This entails that

$$(\varepsilon', \theta_1^*, \dots, \theta_k^*, G(Y, \varepsilon_{k+1}), Z_1, \dots, Z_T)$$

has law μ_{k+1} and setting $\theta_{k+1}^* := G(Y, \varepsilon_{k+1})$ we make sure that θ_{k+1}^* is a function of $\varepsilon', Z_1, \dots, Z_k$, $\varepsilon_1, \dots, \varepsilon_{k+1}$ only, a fortiori, it is $\mathcal{F}_k$-measurable. We finally get all the θ_j^*, $j = 1, \dots, T$ such that the law of

$$(\varepsilon', \theta_1^*, \dots, \theta_T^*, Z_1, \dots, Z_T)$$

equals $\mu = \mu_T$. We will now show that

$$V(X_0; \theta_1^*, \dots, \theta_T^*) \geq \limsup_{j \to \infty} V(X_0; \theta_1(j), \dots, \theta_T(j)), \tag{31}$$

which will conclude the proof.

Indeed, $H_j := X_0 + \sum_{t=1}^T \theta_t(j) \Delta S_t - B$ clearly converges in law to $H := X_0 + \sum_{t=1}^T \theta_t^* \Delta S_t - B$, $j \to \infty$ (note that ΔS_t and B are continuous functions of the Z_t and X_0 is deterministic). By continuity of u_+, u_- also $u_\pm([H_j]_\pm)$ tends to $u_\pm([H]_\pm)$ in law which entails that $P(u_\pm([H_j]_\pm) \geq y) \to P(u_\pm([H]_\pm) \geq y)$ for all y outside a countable set (the points of discontinuities of the cumulative distribution functions of $u_\pm([H]_\pm)$).

It suffices thus to find a measurable function $h(y)$ with $w_+(P(u_+[H_j]_+ \geq y)) \leq h(y), j \geq 1$ and $\int_0^\infty h(y)dy < \infty$ and then Fatou's lemma will imply (31). We get, just like in Lemma 4.4, using Chebishev's inequality, (3) and (5), for $y \geq 1$:

$$\begin{aligned} w_+(P(u_+[H_j]_+ \geq y)) &\leq C\frac{1 + |X_0|^{\alpha\lambda} + \sum_{t=1}^T E\left(|\theta_t(j) - \phi_t|^{\alpha\lambda}|\Delta S_t|^{\alpha\lambda}\right)}{y^{\lambda\gamma}} \\ &\leq \frac{C}{y^{\lambda\gamma}}\left(1 + |X_0|^{\alpha\lambda} + \sum_{t=1}^T E^{1/p}|\theta_t(j) - \phi_t|^{\beta_T} E^{1/q} W_t^q\right), \end{aligned}$$

for some constant $C > 0$ and $W_t \in \mathcal{W}^+$, $t = 1, \ldots, T$, using Hölder's inequality with $p := \beta_T/(\alpha\lambda)$ and its conjugate q (recall that $\Delta S_t \in \mathcal{W}$). We know from the construction that $\sup_{j,t} E|\theta_t(j) - \phi_t|^{\beta_T} < \infty$. Thus we can find some constant $C' > 0$ such that $w_+(P(u_+[H_j]_+ \geq y)) \leq C'/y^{\lambda\gamma}$, for all j. Now trivially $w_+(P(u_+[H_j]_+ \geq y)) \leq w_+(1) = 1$ for $0 \leq y \leq 1$. Setting $h(y) := 1$ for $0 \leq y \leq 1$ and $h(y) := C'/y^{\lambda\gamma}$ for $y > 1$, we conclude since $\lambda\gamma > 1$ and thus $1/y^{\lambda\gamma}$ is integrable on $[1, \infty)$. $\square$

6 Examples

Example 6.1. Let S_0 be constant and $\Delta S_t \in \mathcal{W}$ independent $t = 1, \ldots, T$. Take $Z_i := \Delta S_i$, define $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and $\mathcal{G}_t := \sigma(Z_1, \ldots, Z_t)$, $T \geq 1$. Assume that S_t satisfies (NA) + (R) w.r.t. $\mathcal{G}_t$. Then this continues to hold for the enlargement $\mathcal{F}_t$ defined in Assumption 5.1. So Assumptions 2.2 and 5.1 hold with κ_t, π_t almost surely constants since the conditional law of ΔS_t w.r.t. $\mathcal{F}_{t-1}$ is a.s. equal to its actual law.

Example 6.2. Fix $d \leq N$. Take $Y_0 \in \mathbb{R}^N$ constant and define Y_t by the difference equation

$$Y_{t+1} - Y_t = \mu(Y_t) + \sigma(Y_t)\epsilon_{t+1},$$

where $\mu : \mathbb{R}^N \to \mathbb{R}^N$ is bounded and continuous, $\sigma : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is bounded and continuous. We assume that there is $h > 0$ such that

$$\inf_{|v|=1} v^T\sigma(x)\sigma^T(x)v \geq h$$

for all $x \in \mathbb{R}^N$; $\epsilon_t \in \mathcal{W}$, $t = 1, \ldots, T$ are independent with $\operatorname{supp}\operatorname{Law}\epsilon_t = \mathbb{R}^N$.

Thus Y_t follows a discretized dynamics of a non-degenerate diffusion process. We may think that Y_t represent the evolution of N economic factors. Take $\mathcal{G}_0$ trivial and $\mathcal{G}_t := \sigma(\epsilon_j, j \leq t)$, $t \geq 1$.

We claim that Y_t satisfies Assumption 2.2 with respect to $\mathcal{G}_t$. Indeed, $Y_t \in \mathcal{W}$ is trivial and we will show that (1) holds with κ_t, π_t constants.

Take $v \in \mathbb{R}^N$. Obviously,

$$P(v(Y_{t+1} - Y_t) \leq -|v|/\mathcal{G}_t) = P(v(Y_{t+1} - Y_t) \leq -|v|/Y_t).$$

It is thus enough to show that there is $c > 0$ such that for each unit vector v and each $x \in \mathbb{R}^N$

$$P(v(\mu(x) + \sigma(x)\epsilon_t) \leq -1) \geq c.$$

Denoting by m an upper bound for $|\mu(x)|$, $x \in \mathbb{R}^N$, we may write

$$P(v(\mu(x) + \sigma(x)\epsilon_t) \leq -1) \geq P(v\sigma(x)\epsilon_t \leq -(m+1)).$$

Here $y = v^T\sigma(x)$ is a vector of length at least $\sqrt{h}$, hence the absolute value of one of its components is at least $\sqrt{h/N}$. Thus we have

$$\begin{aligned} P(v^T\sigma(x)\epsilon_t \leq -(m+1)) \quad \geq \quad & \min\Big(\min_{i,k_i} P(\sqrt{h/N}\epsilon_t^i \leq -(m+1), k_i(j)\epsilon_t^j \leq 0,\ j \neq i); \\ & \min_i P(\sqrt{h/N}\epsilon_t^i \geq (m+1), k_i(j)\epsilon_t^j \leq 0,\ j \neq i)\Big) \end{aligned}$$

where i ranges over $1,\dots,N$ and k_i ranges over the (finite) set of all functions from $\{1,2,\dots,i-1,i+1,\dots,N\}$ to $\{1,-1\}$ (representing all the possible configurations for the signs of y^j, $j \neq i$). This minimum is positive by our assumption on the support of ϵ_t.

Now we can take $S_t^i := Y_t^i$, $i = 1,\dots,d$ for some $d \leq N$. When $N > d$, we may think that the Y_j, $d < j \leq N$ are not prices of some traded assets but other relevant economic variables that influence the market. It is easy to check that Assumption 2.2 holds for S_t, too, with respect to $\mathcal{G}_t$.

Enlarging each $\mathcal{G}_t$ by $\tilde{\varepsilon}$, independent of $\epsilon_1,\dots,\epsilon_T$ we get $\mathcal{F}_t$ as in Assumption 5.1. Clearly, Assumption 2.2 continues to hold for S_t with respect to $\mathcal{F}_t$ and Assumption 5.1 is then also true as S_t is a continuous function of $\epsilon_1,\dots,\epsilon_t$.

Example 6.3. Take Y_t as in the above example. For simplicity, we assume $d = N = 1$. Furthermore, let ϵ_t, $t = 1,\dots,T$ be such that for all $\zeta > 0$,

$$Ee^{\zeta|\epsilon_t|} < \infty.$$

Set $S_t := \exp(Y_t)$ this time. We claim that Assumption 2.2 holds true for S_t with respect to the filtration $\mathcal{G}_t$. Obviously, $\Delta S_t \in \mathcal{W}$, $t \geq 1$.

We choose $\kappa_t := S_t/2$. Clearly, $1/\kappa_t \in \mathcal{W}$. It suffices to prove that $1/P(S_{t+1} - S_t \leq -S_t/2/\mathcal{G}_t)$ and $1/P(S_{t+1} - S_t \geq S_t/2/\mathcal{G}_t)$ belong to $\mathcal{W}$. We will show only the second containment, the first one being similar. This amounts to checking

$$1/P(\exp\{Y_{t+1} - Y_t\} \geq 3/2/Y_t) \in \mathcal{W}.$$

Let us notice that

$$\begin{aligned} P(\exp\{Y_{t+1} - Y_t\} \geq 3/2/Y_t) &= P(\mu(Y_t) + \sigma(Y_t)\epsilon_{t+1} \geq \ln(3/2)/Y_t) \\ &= P(\epsilon_{t+1} \geq \frac{\ln(3/2) - \mu(Y_t)}{\sigma(Y_t)}/Y_t) \\ &\geq P(\epsilon_{t+1} \geq \frac{\ln(3/2) - m}{\sqrt{h}}), \end{aligned}$$

which is a deterministic positive constant, by the assumption on the support of ϵ_{t+1}. Defining the enlarged $\mathcal{F}_t$, Assumptions 2.2 and 5.1 hold for S_t. Examples 6.2 and 6.3 are pertinent, in particular, when the ϵ_t are Gaussian.

7 Auxiliary results

We start with a simple observation.

Lemma 7.1. *If $Y \in \mathcal{W}^+$ then*

$$\int_0^\infty P^\delta(Y \geq y)dy < \infty,$$

for all $\delta > 0$.

Proof. As by Chebishev's inequality and $Y \in \mathcal{W}^+$,

$$P(Y \geq y) \leq M(N) y^{-N}, \quad y > 0,$$

for all $N > 0$, for a constant $M(N) := EY^N$, we can choose N so large to have $N\delta > 1$, showing that the integral in question is finite. □

The following Lemmata are fairly standard but we include their proofs in want of an appropriate, precise reference.

Lemma 7.2. *Let E be uniformly distributed on $[0,1]$. Then for each $l \geq 1$ there are measurable $f_1, \ldots, f_l : [0,1] \to [0,1]$ such that $f_1(E), \ldots, f_l(E)$ are independent and uniform on $[0,1]$.*

Proof. We first recall that if $\mathcal{Y}_1, \mathcal{Y}_2$ are uncountable Polish spaces then they are Borel isomorphic, i.e. there is a bijection $\psi : \mathcal{Y}_1 \to \mathcal{Y}_2$ such that ψ, ψ^{-1} are measurable (with respect to the respective Borel fields); see e.g. page 159 of Dellacherie and Meyer [1979].

Fix a Borel-isomorphism $\psi : \mathbb{R} \to [0,1]^l$ and define the probability $\kappa(A) := \lambda_l(\psi(A))$, $A \in \mathcal{B}(\mathbb{R})$, where λ_l is the l-dimensional Lebesgue-measure restricted to $[0,1]^l$. Denote by $F(x) := \kappa((-\infty, x])$, $x \in \mathbb{R}$ the cumulative distribution function (c.d.f.) corresponding to κ and set

$$F^-(u) := \inf\{q \in \mathbb{Q} : F(q) \geq u\}, \ u \in (0,1).$$

This function is measurable and it is well-known that $F^-(E)$ has law κ. Now clearly

$$(f_1(u), \ldots, f_l(u)) := \psi(F^-(u))$$

is such that $(f_1(E), \ldots, f_l(E))$ has law λ_l and the f_i are measurable. □

Lemma 7.3. *Let $\mu(dy, dz) = \nu(y, dz)\delta(dy)$ be a probability on $\mathbb{R}^{N_2} \times \mathbb{R}^{N_1}$ such that $\delta(dy)$ is a probability on $\mathbb{R}^{N_2}$ and $\nu(y, dz)$ is a probabilistic kernel. Assume that Y has law $\delta(dy)$ and E is independent of Y and uniformly distributed on $[0,1]$. Then there is a measurable function $G : \mathbb{R}^{N_2} \times [0,1] \to \mathbb{R}^{N_1}$ such that $(Y, G(Y, E))$ has law $\mu(dy, dz)$.*

Proof. Just like in the previous proof, fix a Borel isomorphism $\psi : \mathbb{R} \to \mathbb{R}^{N_1}$. Consider the measure on $\mathbb{R} \times \mathbb{R}^{N_2}$ defined by $\tilde{\mu}(A \times B) := \int_A \nu(y, \psi(B))\delta(dy)$, $A \in \mathcal{B}(\mathbb{R}^{N_2})$, $B \in \mathcal{B}(\mathbb{R})$. For δ-almost every y, $\nu(y, \psi(\cdot))$ is a probability measure on $\mathbb{R}$. Let $F(y, z) := \nu(y, \psi((-\infty, z]))$ denote its cumulative distribution function and define

$$F^-(y, u) := \inf\{q \in \mathbb{Q} : F(y, q) \geq u\}, \ u \in (0,1),$$

this is easily seen to be $\mathcal{B}(\mathbb{R}^{N_2}) \otimes \mathcal{B}([0,1])$-measurable. Then, for δ-almost every y, $F^-(y, E)$ has law $\nu(y, \psi(\cdot))$. Hence $(Y, F^-(Y, E))$ has law $\tilde{\mu}$. Consequently, $(Y, \psi(F^-(Y, E)))$ has law μ and we may conclude setting $G(y, u) := \psi(F^-(y, u))$. The technique of this proof is well-known, see e.g. page 228 of Bhattacharya and Waymire [1990]. □

Lemma 7.4. *Let $(\tilde{Z}, W)$ be a $\mathbb{R} \times \mathbb{R}^k$-valued random variable with continuous everywhere positive density $f(x^1, \ldots, x^{k+1})$ (with respect to the $k+1$-dimensional Lebesgue measure) such that the function*

$$x^1 \to \sup_{x^2, \ldots, x^k} f(x^1, \ldots, x^{k+1}) \tag{32}$$

is integrable on $\mathbb{R}$. Then there is a homeomorphism $H : \mathbb{R}^{k+1} \to [0,1] \times \mathbb{R}^k$ such that $H^i(x^1, \ldots, x^{k+1}) = x^i$ for $i = 2, \ldots, k+1$ and $Z := H^1(\tilde{Z}, W)$ is uniform on $[0,1]$, independent of W.

Proof. The conditional distribution function of $\tilde{Z}$ knowing $W = (x^2, \dots, x^{k+1})$,

$$F(x^1, \dots, x^{k+1}) := \frac{\int_{-\infty}^{x^1} f(z, x^2, \dots, x^{k+1})dz}{\int_{-\infty}^{\infty} f(z, x^2, \dots, x^{k+1})dz},$$

is continuous (due to the integrability of (32) and Lebesgue's theorem). By everywhere positivity of f, F is also strictly increasing in x^1. It follows that the function

$$H : (x^1, \dots, x^{k+1}) \to (F(x^1, \dots, x^{k+1}), x^2, \dots, x^{k+1})$$

is a bijection and hence a homeomorhpism by Theorem 4.3 in Deimling [1985]. The conditional law $P(H^1(\tilde{Z}, W) \in \cdot | W = (x^2, \dots, x^{k+1}))$ is uniform on $[0,1]$ for Lebesgue-almost all $(x^2, \dots, x^{k+1})$, which shows that $H^1(\tilde{Z}, W)$ is independent of W with uniform law on $[0,1]$. □

Corollary 7.5. *Let $(\tilde{Z}_1, \dots, \tilde{Z}_k)$ be an $\mathbb{R}^k$-valued random variable with continuous and everywhere positive density (w.r.t. the k-dimensional Lebesgue measure) such that for all $i = 1, \dots, k$, the function*

$$z \to \sup_{x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^k} f(x^1, \dots, x^{i-1}, z, x^{i+1}, \dots, x^k) \tag{33}$$

is integrable on $\mathbb{R}$. There are independent random variables $Z_1, \dots, Z_k$ and homeomorphisms $g_l(k) : \mathbb{R}^l \to \mathbb{R}^l$, $1 \le l \le k$ such that $(\tilde{Z}_1, \dots, \tilde{Z}_l) = g_l(k)(Z_1, \dots, Z_l)$.

Proof. The case $k = 1$ is vacuous and $k = 2$ follows by Lemma 7.4. Assume that the statement is true for k, let us prove it for $k + 1$. We may set $g_l(k+1) := g_l(k)$, $1 \le l \le k$, it remains to construct $g_{k+1}(k+1)$ and Z_{k+1}.

Lemma 7.4 provides a homeomorphism $s : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ such that $s^m(x^1, \dots, x^{k+1}) = x^m$, $1 \le m \le k$ and $Z_{k+1} := s^{k+1}(\tilde{Z}_1, \dots, \tilde{Z}_{k+1})$ is independent of $(\tilde{Z}_1, \dots, \tilde{Z}_k)$ (and hence of $(Z_1, \dots, Z_k) = g_k(k)^{-1}(\tilde{Z}_1, \dots, \tilde{Z}_k)$). Define $a : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ by

$$\begin{aligned} a(x^1, \dots, x^{k+1}) &:= (g_k(k)^{-1}(x^1, \dots, x^k), s^{k+1}(x^1, \dots, x^{k+1})) \\ &= s(g_k(k)^{-1}(x^1, \dots, x^k), x^{k+1}), \end{aligned}$$

a is a homeomorphism since it is the composition of two homeomorphisms. As then $a(\tilde{Z}_1, \dots, \tilde{Z}_{k+1}) = (Z_1, \dots, Z_{k+1})$, the result is shown setting $g_{k+1}(k+1) := a^{-1}$. □

Remark 7.6. Condition (33) is quite weak, it holds, for example, when there is $C > 0$ such that

$$f(\mathbf{x}) \le \frac{C}{1 + |\mathbf{x}|^2}, \quad \mathbf{x} \in \mathbb{R}^k.$$

Assumption 7.7. Let $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $\mathcal{G}_t = \sigma(\tilde{Z}_1, \dots, \tilde{Z}_t)$ for $1 \le t \le T$, where the $\tilde{Z}_i$, $i = 1, \dots, T$ are $\mathbb{R}^N$-valued random variables with a continuous and everywhere positive joint density f on $\mathbb{R}^{TN}$ such that (33) holds with $k = TN$. S_0 is constant and ΔS_t is a continuous function of $(\tilde{Z}_1, \dots, \tilde{Z}_t)$, for all $t \ge 1$.

Furthermore, $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{F}_0$, $t \ge 0$, where $\mathcal{F}_0 = \sigma(\tilde{\varepsilon})$ with $\tilde{\varepsilon}$ uniformly distributed on $[0,1]$ and independent of $(\tilde{Z}_1, \dots, \tilde{Z}_T)$.

Remark 7.8. If Assumption 7.7 above holds true then so does Assumption 5.1, in virtue of Corollary 7.5. Assumption 7.7 may be easier to verify in certain model classes.

References

A.B. Berkelaar, R. Kouwenberg, and T. Post. Optimal portfolio choice under loss aversion. *Rev. Econ. Stat.*, 86:973–987, 2004.

C. Bernard and M. Ghossoub. Static portfolio choice under cumulative prospect theory. *Mathematics and Financial Economics*, 2:277–306, 2010.

R. N. Bhattacharya and E. C. Waymire. *Stochastic Processes with Applications.* John Wiley and Sons, New York, 1990.

L. Carassus and H. Pham. Portfolio optimization for nonconvex criteria functions. *RIMS Kôkyuroku series, ed. Shigeyoshi Ogawa*, 1620:81–111, 2009.

L. Carassus and M. Rásonyi. Optimal strategies and utility-based prices converge when agents' preferences do. *Math. Oper. Res.*, 32:102–117, 2007.

G. Carlier and R.-A. Dana. Optimal demand for contingent claims when agents have law invariant utilities. *Math. Finance*, 21:169–201, 2011.

R.C. Dalang, A. Morton, and W. Willinger. Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stochastics Stochastics Rep.*, 29(2):185–201, 1990.

K. Deimling. *Nonlinear functional analysis.* Springer-Verlag, Berlin, 1985.

C. Dellacherie and P.-A. Meyer. *Probability and potential.* North-Holland, Amsterdam, 1979.

H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time.* Walter de Gruyter & Co., Berlin, 2002.

X. He and X.Y. Zhou. Portfolio choice under cumulative prospect theory: An analytical treatment. *Forthcoming in Management Science.*

H. Jin and X.Y. Zhou. Behavioural portfolio selection in continuous time. *Math. Finance*, 18: 385–426, 2008.

D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 47:263–291, 1979.

A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk & Uncertainty*, 5:297–323, 1992.